Notation

Numbers and Arrays

x	A scalar, or scalar function $x(\cdot)$
$\boldsymbol{x}$	A vector, or vector-valued function $\boldsymbol{x}(\cdot)$
$\boldsymbol{x}_1 \oplus \boldsymbol{x}_2$	Direct sum (stacking) of vectors, such that $\boldsymbol{x}_1 \oplus \boldsymbol{x}_2 := \left[\begin{smallmatrix}\boldsymbol{x}_1 \\ \boldsymbol{x}_2\end{smallmatrix}\right]$
$\boldsymbol{K}$	A matrix
$\boldsymbol{A} \oplus \boldsymbol{B}$	Direct sum of matrices, such that $\boldsymbol{A} \oplus \boldsymbol{B} := \left[\begin{smallmatrix}\boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{B}\end{smallmatrix}\right]$
K	A linear operator
$\mathsf{A} \oplus \mathsf{B}$	Direct sum of linear operators, such that $\mathsf{A} \oplus \mathsf{B} := \left[\begin{smallmatrix}\mathsf{A} & \mathsf{O} \\ \mathsf{O} & \mathsf{B}\end{smallmatrix}\right]$
$\boldsymbol{I}$	Identity matrix
I	Identity operator

Sets, Vector Spaces, and Function spaces

$\mathcal{X}, \mathcal{Z}, \mathcal{H}, \mathcal{F}$	A vector/Hilbert space
$\mathbb{I}_{\mathcal{X}}$	A basis set of the vector space $\mathcal{X}$
$\mathbb{R}, \mathbb{C}$	The set of real and complex numbers
$\mathcal{X} \oplus \mathcal{Y}$	Direct sum of vector spaces $\mathcal{X}$ and $\mathcal{Y}$ such that if $\boldsymbol{x} \in \mathcal{X}$ and $\boldsymbol{y} \in \mathcal{Y}$, then $\boldsymbol{x} \oplus \boldsymbol{y} \in \mathcal{X} \oplus \mathcal{Y}$
$\mathcal{F}$	A function space
$f_{\boldsymbol{\alpha}} \in \mathcal{F}$	A function in the function space $\mathcal{F}$, represented with the coefficients $\boldsymbol{\alpha}$ on a chosen basis $\mathbb{I}_{\mathcal{F}} = \{\hat{f}_1, \dots\}$, such that $f_{\boldsymbol{\alpha}}(\cdot) = \sum_{i=1}^{m} \alpha_i \hat{f}_i(\cdot)$, given $\boldsymbol{\alpha} = [\alpha_1, \dots]$.

Group and representation theory

$\mathbb{G}$	A symmetry group
g, g_1, g_a	A symmetry group element
$g \triangleright \boldsymbol{x}$	The (left) group action of g on $\boldsymbol{x}$ defined by $g \triangleright \boldsymbol{x} := \rho_{\mathcal{X}}(g)\mathcal{X}$, for a chosen basis $\mathbb{I}_{\mathcal{X}}$
$\rho_{\mathcal{X}}$	A representation of the group $\mathbb{G}$ on the vector space $\mathcal{X}$, defined for a chosen basis $\mathbb{I}_{\mathcal{X}}$
$\rho_{\mathcal{X}}(g)$	Representation of the group element g on the vector space $\mathcal{X}$, defined for a chosen basis $\mathbb{I}_{\mathcal{X}}$
$\rho_{\mathcal{X}} \oplus \rho_{\mathcal{Y}}$	Direct sum of group representations, such that $\rho_{\mathcal{X}} \oplus \rho_{\mathcal{Y}} := \left[\begin{smallmatrix}\rho_{\mathcal{X}} & \\ & \rho_{\mathcal{Y}}\end{smallmatrix}\right]$
$\mathbb{G}\boldsymbol{x}$	The group orbit of $\boldsymbol{x}$, defined as $\mathbb{G}\boldsymbol{x} := \{g \triangleright \boldsymbol{x} \mid g \in \mathbb{G}\}$
$\mathbb{G}_a \times \mathbb{G}_b$	Direct product of groups $\mathbb{G}_a$ and $\mathbb{G}_b$
$\mathbb{U}(\mathcal{X})$	Unitary group on the vector space $\mathcal{X}$
$\mathbb{GL}(\mathcal{X})$	General Linear group on the vector space $\mathcal{X}$
$\mathbb{C}_n$	Cyclic group of order n
$\mathbb{K}_4$	Klein four-group

Appendix I. Appendix

I.1. Finite dimensional function spaces of observable functions

Given a set of observable functions $\mathbb{I}_{\mathcal{X}} = \{x_1, \dots, x_m \mid x_i : \Omega \mapsto \mathbb{R}, \forall i \in [1, m]\}$, we can interprete these functions as the components of a vector-valued function $\boldsymbol{x} = [x_1, \dots] : \Omega \mapsto \mathbb{R}^m$, that enables the numerical representation of the state ω as a point in a finite-dimensional vector space $\boldsymbol{x}(\omega) \in \mathcal{X} \subseteq \mathbb{R}^m$. For the objective of our work, we will also interprete $\mathbb{I}_{\mathcal{X}}$ as the basis set of a finite-dimensional function space $\mathcal{F}_{\mathcal{X}} : \Omega \mapsto \mathbb{R}$, such that any observable function $x \in \mathcal{F}_{\mathcal{X}}$ is defined by the linear combination of the basis functions $x_{\boldsymbol{\alpha}}(\omega) := \langle \boldsymbol{x}(\,\cdot\,), \boldsymbol{\alpha} \rangle = \sum_{i=1}^{m} \alpha_i x_i(\omega) = \boldsymbol{x}(\omega)^{\intercal}\boldsymbol{\alpha}$. Where $\boldsymbol{\alpha} = [\alpha_1, \dots] \in \mathbb{R}^m$ are the coefficients of x in the basis of $\mathcal{F}_{\mathcal{X}}$, and the notation $x_{\boldsymbol{\alpha}}(\,\cdot\,)$ highlight the relationship between the function x and its coefficient vector representation $\boldsymbol{\alpha}$.

I.1.1. SYMMETRIES OF THE STATE REPRESENTATION

When the dynamical system possess a state symmetry group $\mathbb{G}$ (def. 1), appropriate numerical representations of the state are constrained to be $\mathbb{G}$-equivariant vector-value functions (see eq. (8) and prop. 2). This ensures that the symmetry relationship between any state $\omega \in \Omega$ and its symmetric states $\mathbb{G}\omega := \{g \triangleright \omega \mid g \in \mathbb{G}\}$ is preserved in the representation space $\mathcal{X}$, such that $\mathbb{G}\boldsymbol{x}(\omega) = \{g \triangleright \boldsymbol{x}(\omega) \mid g \in \mathbb{G}\} \subset \mathcal{X}$.

$$
\begin{array}{ccc}
\omega & \xrightarrow{\;g \in \mathbb{G}\;} & g \triangleright \omega \\
\big\downarrow \boldsymbol{x} & & \big\downarrow \boldsymbol{x} \\
\boldsymbol{x}(\omega) & \xrightarrow{\;g \in \mathbb{G}\;} & g \triangleright \boldsymbol{x}(\omega)
\end{array}
\tag{8}
$$

I.1.2. SYMMETRIC FUNCTION SPACES

When $\mathcal{X}$ is a $\mathbb{G}$-symmetric space, the group is defined to act on any chosen basis set of the space, including the observable functions $\mathbb{I}_{\mathcal{X}}$. This, in turn, ensures that the finite-dimensional function space $\text{span}(\mathbb{I}_{\mathcal{X}}) := \mathcal{F}_{\mathcal{X}} : \Omega \mapsto \mathbb{R}$ features the symmtry group $\mathbb{G}$, being the elements of the space $\mathbb{G}$-equivariant functions, i.e., $\mathcal{F}_{\mathcal{X}} = \{x \mid g \triangleright x_{\boldsymbol{\alpha}}(\omega) = x_{\boldsymbol{\alpha}}(g^{-1} \triangleright \omega) = x_{g \triangleright \boldsymbol{\alpha}}(\omega), \forall g \in \mathbb{G}\}$ (see def. 2). Where the notation $g \triangleright x_{\boldsymbol{\alpha}}(\omega) = x_{g \triangleright \boldsymbol{\alpha}}(\omega)$ describes the action of a symmetry transformation on a observable function, as a linear transformation on its coefficients vector representation $\boldsymbol{\alpha}$.

I.2. Group and representation theory

Definition 2 (Group action on a function space) *The (left) action of a group $\mathbb{G}$ on the space of functions $\mathcal{X} : \Omega \to \mathbb{C}$, where Ω is a set with symmetry group $\mathbb{G}$, is defined as:*

$$
\begin{aligned}
(\triangleright) : \quad \mathbb{G} \times \mathcal{X} &\longrightarrow \mathcal{X} \\
(g, x(\omega)) &\longrightarrow g \triangleright x(\omega) \doteq x(g^{-1} \triangleright \omega)
\end{aligned}
\tag{9a}
$$

From an algebraic perspective, the action inversion (contragredient representation) emerges to ensure that the symmetry group in the function space is a homomorphism of the group in the domain $(g_1 \triangleright g_2) \triangleright x(\omega) \doteq x((g_1 \triangleright g_2)^{-1} \triangleright \omega)$. Which can be proven by a couple of algebraic steps:

$$
(g_1 \triangleright (g_2 \triangleright x))(\omega) = (g_1 \triangleright x_{g_2})(\omega) = g_2 \triangleright x(g_1^{-1}\omega) = x((g_2^{-1} \triangleright g_1^{-1}) \triangleright \omega) = x((g_1 \triangleright g_2)^{-1} \triangleright \omega)
\tag{9b}
$$

From a geometric perspective, when $\mathcal{X}$ is a separable Hilbert space, each function can be associated with its vector of coefficients representation $x_{\boldsymbol{\alpha}}(\,\cdot\,) := \sum_{i=1}^{m} \alpha_i x_i(\,\cdot\,) = \boldsymbol{x}(\,\cdot\,)^{\intercal}\boldsymbol{\alpha}$. Here, $\boldsymbol{x} = [x_1, \dots]$ represents the basis functions of $\mathcal{X}$. As the function space is symmetric, the group $\mathbb{G}$ acts on the basis set, leading to a group representation acting on the basis functions $g \triangleright \boldsymbol{x}(\,\cdot\,) = \rho_{\mathcal{X}}(g)\boldsymbol{x}(\,\cdot\,)$. The unitary representation of the group $\mathbb{G}$ on the function space is denoted by $\rho_{\mathcal{X}} : \mathbb{G} \to \mathbb{U}(\mathcal{X})$, which is an invertible matrix/operator. This representation enable us to interpret the *symmetry transformations of the function space as point transformations, where the points are the function's coefficient vector representation* $\boldsymbol{\alpha}$*, that is:*

$$\begin{aligned} g \triangleright x_{\boldsymbol{\alpha}}(\,\cdot\,) &:= \sum_{i=1}^{m} \alpha_i x_i(g^{-1} \triangleright \,\cdot\,) \\ &= (\boldsymbol{x}(g^{-1} \triangleright \,\cdot\,))^{\intercal}\boldsymbol{\alpha} \\ &= (g^{-1} \triangleright \boldsymbol{x}(\,\cdot\,))^{\intercal}\boldsymbol{\alpha} \\ &= \boldsymbol{x}(\,\cdot\,)^{\intercal} g \triangleright \boldsymbol{\alpha} \\ &= x_{g \triangleright \boldsymbol{\alpha}}(\,\cdot\,) \end{aligned} \tag{10}$$

Lemma 1 (Schur's Lemma for Unitary representations (Knapp, 1986, Prop 1.5)) *Consider two Hilbert spaces,* $\mathcal{X}$ *and* $\mathcal{X}'$*, each with their respective irreducible unitary representations, denoted as* $\bar{\rho}_{\mathcal{X}} : \mathbb{G} \to \mathbb{U}(\mathcal{X})$ *and* $\bar{\rho}_{\mathcal{X}'} : \mathbb{G} \to \mathbb{U}(\mathcal{X}')$*. Suppose* $\mathsf{T} : \mathcal{X} \to \mathcal{X}'$ *is a linear equivariant operator such that* $\bar{\rho}_{\mathcal{X}'}\mathsf{T} = \mathsf{T}\bar{\rho}_{\mathcal{X}}$*. If the irreducible representations are not equivalent, i.e.,* $\bar{\rho}_{\mathcal{X}} \nsim \bar{\rho}_{\mathcal{X}'}$*, then* T *is the trivial (or zero) map. Conversely, if* $\bar{\rho}_{\mathcal{X}} \sim \bar{\rho}_{\mathcal{X}'}$*, then* T *is a constant multiple of an isomorphism (def. 4). Denoting* I *as the identity operator, this can be expressed as:*

$$\bar{\rho}_{\mathcal{X}} \nsim \bar{\rho}_{\mathcal{X}'} \iff \mathbf{0}_{\mathcal{X}'} = \mathsf{T}\boldsymbol{h} \mid \forall\, \boldsymbol{h} \in \mathcal{X} \tag{11a}$$

$$\bar{\rho}_{\mathcal{X}} \sim \bar{\rho}_{\mathcal{X}'} \iff \mathsf{T} = \alpha\mathsf{U} \mid \alpha \in \mathbb{C}, \mathsf{U} \cdot \mathsf{U}^{H} = \mathsf{I} \tag{11b}$$

$$\bar{\rho}_{\mathcal{X}} = \bar{\rho}_{\mathcal{X}'} \iff \mathsf{T} = \alpha\mathsf{I} \tag{11c}$$

For intiution refeer to the following blog post

Definition 3 (Group stable space & Group irreducible stable spaces) *Let* $\rho_{\mathcal{X}} : \mathbb{G} \to \mathbb{U}(\mathcal{X})$ *be a unitary representation on the Hilbert space* $\mathcal{X}$*. A subspace* $\mathcal{X}' \subseteq \mathcal{X}$ *is said to be* $\mathbb{G}$*-stable if*

$$\rho_{\mathcal{X}}(g)\boldsymbol{h} \in \mathcal{X}' \quad \mid \boldsymbol{h} \in \mathcal{X}' \quad \forall \quad \boldsymbol{w} \in W, g \in \mathbb{G}. \tag{12}$$

If the only $\mathbb{G}$*-stable subspaces of* $\mathcal{X}'$ *are* $\mathcal{X}'$ *itself and* $\{\mathbf{0}\}$*, the space is said to be an irreducible* $\mathbb{G}$*-stable space.*

Definition 4 (Homomorphism, Isomorphism and equivariant linear maps) *Let* $\mathbb{G}$ *be a symmetry group and* $\mathcal{X}$ *and* $\mathcal{X}'$ *be two distinct symmetric Hilbert spaces endowed with unitary representations* $\rho_{\mathcal{X}} : \mathbb{G} \to \mathbb{U}(\mathcal{X})$ *and* $\rho_{\mathcal{X}'} : \mathbb{G} \to \mathbb{U}(\mathcal{X}')$*, respectively.*

A linear map $\mathsf{T} : \mathcal{X} \to \mathcal{X}'$ *is said to be* $\mathbb{G}$*-equivariant if it commutes with the group representations:* $\rho_{\mathcal{X}'}(g)\mathsf{T} = \mathsf{T}\rho_{\mathcal{X}}(g) \mid \forall\, g \in \mathbb{G}$*. The space of all* $\mathbb{G}$*-equivariant linear maps is refered to as the space of homomorphisms (structure preserving maps) and its denoted as* $\mathit{Homo}_{\mathbb{G}}(\mathcal{X}, \mathcal{X}')$ *The spaces are said to be isomorphic if the* $\mathbb{G}$*-equivariant map is invertible. The space of all invertible* $\mathbb{G}$*-equivariant linear maps between* $\mathcal{X}$ *and* $\mathcal{X}'$ *is denoted as* $\mathit{Iso}_{\mathbb{G}}(\mathcal{X}, \mathcal{X}') \subset \mathit{Homo}_{\mathbb{G}}(\mathcal{X}, \mathcal{X}')$*.*

Graphically, the diagrams of a homomorphism and isomorphism between $\mathcal{X}$ *and* $\mathcal{X}'$ *are:*

$$\underbrace{\begin{array}{ccc} \mathcal{X} & \xrightarrow{\rho_{\mathcal{X}}} & \mathcal{X} \\ \downarrow{\scriptstyle \mathsf{T}} & & \downarrow{\scriptstyle \mathsf{T}} \\ \mathcal{X}' & \xrightarrow{\rho_{\mathcal{X}}} & \mathcal{X}' \end{array}}_{Homomorphism} \quad \mathsf{T} \in \mathit{Homo}_{\mathbb{G}}(\mathcal{X}, \mathcal{X}') \qquad \underbrace{\begin{array}{ccc} \mathcal{X} & \xrightarrow{\rho_{\mathcal{X}}} & \mathcal{X} \\ \updownarrow{\scriptstyle \mathsf{T}} & & \updownarrow{\scriptstyle \mathsf{T}} \\ \mathcal{X}' & \xrightarrow{\rho_{\mathcal{X}'}} & \mathcal{X}' \end{array}}_{Isomorphism} \quad \mathsf{T} \in \mathit{Iso}_{\mathbb{G}}(\mathcal{X}, \mathcal{X}') \tag{13}$$